Some Properties of the Elliptic Ordinary Differential Equation

Shikuo Liu[a], Zuntao Fu[a,b], Shida Liu[a,b], and Zhanggui Wang[c]

[a] School of Physics, Peking University, Beijing, 100871, China
[b] STCS, Peking University, Beijing, 100871, China
[c] National Marine Environmental Predicting Center, SOA, Beijing, 100081, China

Reprint requests to Dr. Z. F.; E-mail: fuzt@pku.edu.cn

Z. Naturforsch. **60a,** 566 – 572 (2005); received June 10, 2005

In this paper, some properties of the elliptic ordinary differential equation, which can be used to find travelling wave solutions of nonlinear evolution equations, are given. – PACS: 03.65.Ge

Key words: Elliptic Ordinary Differential Equation; Nonlinear Evolution Equations.

1. Introduction

The elliptic ordinary differential equation (EODE) plays an important role in finding travelling wave solutions of nonlinear evolution equations. By means of the EODE, various periodic solutions and solitary wave solutions can be obtained [1 – 9]. Furthermore, if more properties of EODE are found, further solutions of nonlinear evolution equations can be more easily derived. In the next sections, we will show some properties of EODE, where many examples are used to illustrate these properties.

2. The Elliptic Ordinary Differential Equation

In order to find travelling wave solutions, which include the periodic and solitary wave solutions, the nonlinear partial differential equations (NPDE), especially the nonlinear evolution equations, are often reduced to the following EODE [10]

$$y'^2 = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4, \quad (1)$$

where the prime denotes $\mathrm{d}/\mathrm{d}\xi$.

For example, the Korteweg - de Vries (KdV) equation reads

$$u_t + u u_x + \beta u_{xxx} = 0. \quad (2)$$

We seek its travelling wave solutions of the form

$$u = u(\xi), \quad \xi = x - ct. \quad (3)$$

Then (2) reduces to

$$\beta \frac{\mathrm{d}^3 u}{\mathrm{d}\xi^3} + u\frac{\mathrm{d}u}{\mathrm{d}\xi} - c\frac{\mathrm{d}u}{\mathrm{d}\xi} = 0. \quad (4)$$

Integrating (4) with respect to ξ twice yields

$$u'^2 = -\frac{1}{3\beta}(u^3 - 3cu^2 - 6Au - 6B), \quad (5)$$

with two integration constants A and B. Obviously, (5) is a special kind of (1) with $a_0 = \frac{2B}{\beta}$, $a_1 = \frac{2A}{\beta}$, $a_2 = \frac{c}{\beta}$, $a_3 = -\frac{1}{3\beta}$ and $a_4 = 0$.

Similarly, the combined Korteweg - de Vries - modified - Korteweg - de Vries (KdV - mKdV) equation

$$u_t + u u_x + \alpha u^2 u_x + \beta u_{xxx} = 0 \quad (6)$$

can be rewritten as

$$u'^2 = -\frac{\alpha}{6\beta}[u^4 + \frac{2\beta}{\alpha}u^3 - \frac{6c}{\alpha}u^2 - \frac{12A}{\alpha}u - \frac{12B}{\alpha}], \quad (7)$$

where again A and B are two integration constants. Obviously, (7) is another special kind of (1) with $a_0 = \frac{2B}{\beta}$, $a_1 = \frac{2A}{\beta}$, $a_2 = \frac{c}{\beta}$, $a_3 = -\frac{1}{3\beta}$ and $a_4 = -\frac{\alpha}{6\beta}$.

3. EODE of the First Kind

Considering $a_1 = a_3 = 0$ and $a_0 \neq 0$, $a_2 \neq 0$, $a_4 \neq 0$, then (1) reduces to

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4, \quad (8)$$

which may be called EODE of the first kind.

Next, we will show that there exist some interesting properties.

Property 1. If y is a solution to (8), then $z = \frac{1}{y}$ satisfies

$$z'^2 = a_4 + a_2 z^2 + a_0 z^4, \quad (9)$$

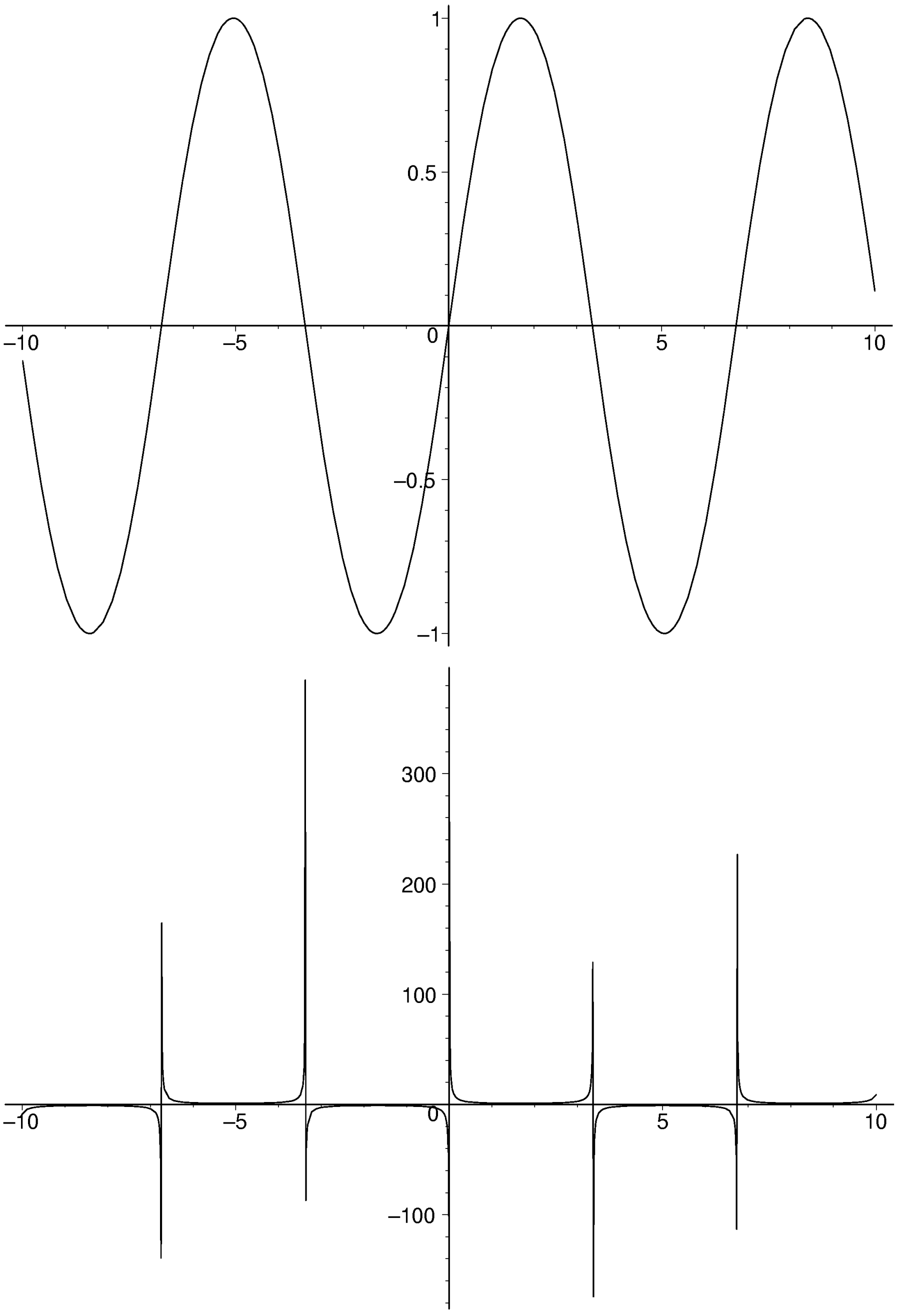

Fig. 1. Graphical presentations of the solutions $y = \mathrm{sn}(t,m)$ (top) and $z = \mathrm{ns}(t,m)$ (bottom) for $m = 0.5$.

that is to say, (9) has the same form as (8) with only a_0 exchanged by a_4, so both y and $\frac{1}{y}$ are solutions of (8) with $a_0 = a_4$ and $a_4 = a_0$.

From this property, we can easily derive a solution to (8) from another solution. Actually, from these two solutions we can also construct more solutions based on the possibility to combine travelling wave solutions to the nonlinear equation [11]. This can be illustrated by some examples next.

Example 1. The equation

$$y'^2 = 1 - (1+m^2)y^2 + m^2y^4, \quad 0 \le m \le 1, \quad (10)$$

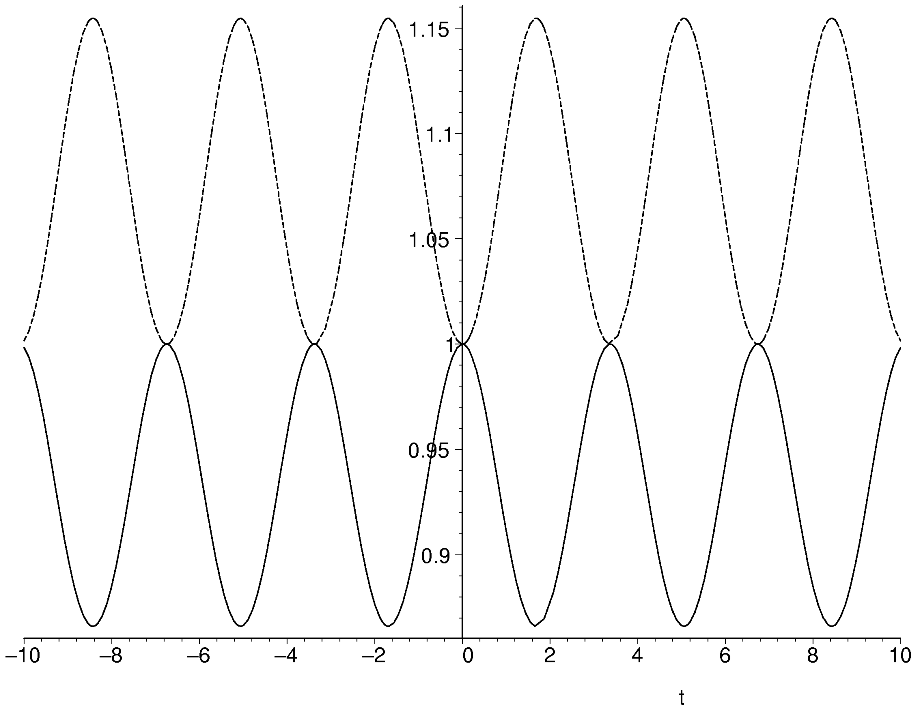

Fig. 2. Graphical presentations of the solutions $y = \mathrm{dn}(t,m)$ (solid line) and $z = \mathrm{nd}(t,m)$ (dashed line) for $m = 0.5$.

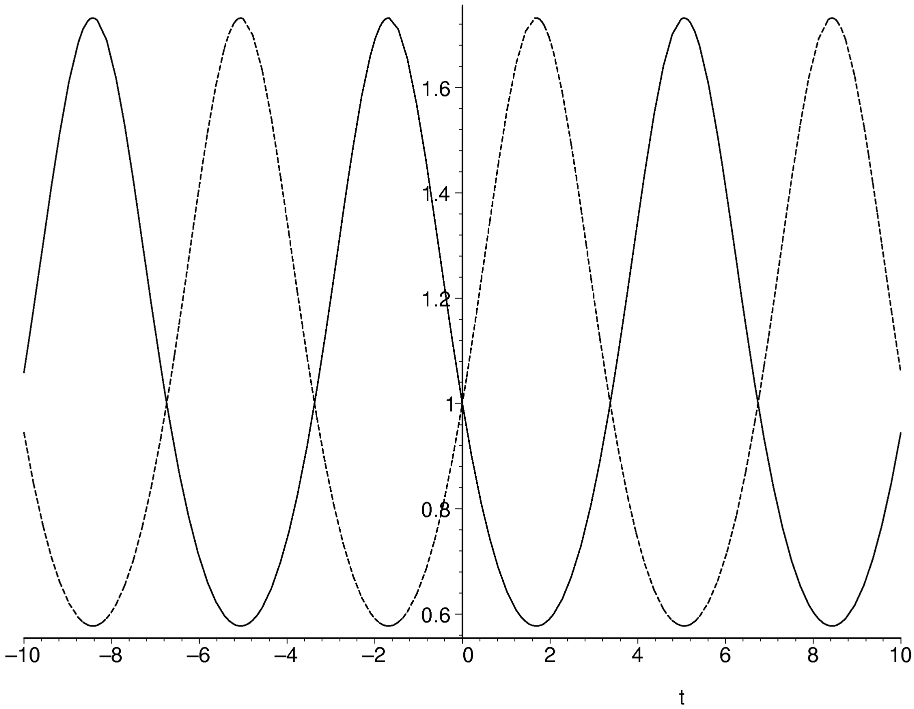

Fig. 3. Graphical presentations for solutions $y_1 = \frac{\mathrm{dn}(t,m)}{1+m\,\mathrm{sn}(t,m)}$ (solid line) and $y_2 = \frac{1+m\,\mathrm{sn}(t,m)}{\mathrm{dn}(t,m)}$ (dashed line) for $m = 0.5$.

has a solution

$$y = \mathrm{sn}(\xi, m). \qquad (11)$$

Then equation

$$z'^2 = m^2 - (1+m^2)z^2 + z^4, \quad 0 \le m \le 1, \qquad (12)$$

has a solution

$$z = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}, \tag{13}$$

where $\mathrm{sn}(\xi, m)$ is the Jacobi elliptic sine function with the modulus m [10, 12–14].

The graphical presentations of y (top) and z (bottom) with $m = 0.5$ for different arguments t are shown in Figure 1. It is obvious that the graphical presentations of y and z show different behaviors, y is bounded but z is sometimes blowup, both of them are periodic argument t.

Example 2. Similarly, $y = \mathrm{cn}(\xi, m)$ is a solution to equation

$$y'^2 = (1 - m^2) + (2m^2 - 1)y^2 - m^2y^4, \tag{14}$$

then $z = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}$ is a solution to equation

$$z'^2 = -m^2 + (2m^2 - 1)z^2 + (1 - m^2)z^4, \tag{15}$$

where $\mathrm{cn}(\xi, m)$ is the Jacobi elliptic cosine function [10, 12–14].

Example 3. If $y = \mathrm{dn}(\xi, m)$ is a solution to equation

$$y'^2 = -(1 - m^2) + (2 - m^2)y^2 - y^4, \tag{16}$$

then $z = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}$ is also a solution to equation

$$z'^2 = -1 + (2 - m^2)z^2 - (1 - m^2)z^4, \tag{17}$$

where $\mathrm{dn}(\xi, m)$ is the Jacobi elliptic function of the third kind [10, 12–14].

Taking the solutions $y = \mathrm{dn}(t, m)$ and $z = \mathrm{nd}(t, m)$ in Example 3, the graphical presentations of y (solid line) and z (dashed line) with $m = 0.5$ for different arguments t are shown in Figure 2. It is obvious that the graphical presentations of y and z show mirror-symmetry and have the same period and amplitude but inverse phases.

Example 4. The equation

$$y'^2 = \frac{1 - m^2}{4} + \frac{1 + m^2}{2}y^2 + \frac{1 - m^2}{4}y^4, \tag{18}$$

has the solution

$$y_1 = \frac{\mathrm{cn}(\xi, m)}{1 + \mathrm{sn}(\xi, m)} = \frac{1 - \mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{19}$$

Since $a_0 = a_4$, it must have another solution

$$y_2 = \frac{\mathrm{cn}(\xi, m)}{1 - \mathrm{sn}(\xi, m)} = \frac{1 + \mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{20}$$

Example 5. Similarly, the equation

$$y'^2 = -\frac{1 - m^2}{4} + \frac{1 + m^2}{2}y^2 - \frac{1 - m^2}{4}y^4 \tag{21}$$

has the two solutions

$$\begin{aligned} y_1 &= \frac{\mathrm{dn}(\xi, m)}{1 + m\mathrm{sn}(\xi, m)} = \frac{1 - m\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \\ y_2 &= \frac{\mathrm{dn}(\xi, m)}{1 - m\mathrm{sn}(\xi, m)} = \frac{1 + m\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \end{aligned} \tag{22}$$

The graphical presentations of y_1 (solid line) and y_2 (dashed line) with $m = 0.5$ for different arguments t are shown in Figure 3. It is obvious that the graphical presentations of y_1 and y_2 show rotational symmetry and have the same period and amplitude but 180 degree rotational phase difference.

Example 6. The equation

$$y'^2 = \frac{m^2}{4} - \frac{2 - m^2}{2}y^2 + \frac{m^2}{4}y^4 \tag{23}$$

has the following two solutions

$$\begin{aligned} y_1 &= \frac{m\mathrm{sn}(\xi, m)}{1 + \mathrm{dn}(\xi, m)} = \frac{1 - \mathrm{dn}(\xi, m)}{m\mathrm{sn}(\xi, m)}, \\ y_2 &= \frac{m\mathrm{sn}(\xi, m)}{1 - \mathrm{dn}(\xi, m)} = \frac{1 + \mathrm{dn}(\xi, m)}{m\mathrm{sn}(\xi, m)}. \end{aligned} \tag{24}$$

Property 2. If Ay_1, with A a nonzero constant, is a solution to (8), then $z = \frac{A}{y_1}$ satisfies

$$z'^2 = a_4A^4 + a_2z^2 + \frac{a_0}{A^4}z^4, \tag{25}$$

so A/y_1 is another solution to (8) with $a_0 = a_4A^4$ and $a_4 = a_0/A^4$. Based on this property, we can obtain more solutions to nonlinear equations, which is demonstrated below by some examples.

Example 7. The equation

$$y'^2 = A^2 - (1 + m^2)y^2 + \frac{m^2}{A^2}y^4 \tag{26}$$

has a solution

$$y = Ay_1 = A \cdot \mathrm{sn}(\xi, m); \tag{27}$$

then the equation

$$z'^2 = m^2A^2 - (1 + m^2)z^2 + \frac{1}{A^2}z^4 \tag{28}$$

has a solution

$$z = A \cdot \mathrm{ns}(\xi, m). \tag{29}$$

Similarly, further examples of this type can be obtained with $A \cdot \mathrm{cn}(\xi, m)$, and $A \cdot \mathrm{dn}(\xi, m)$.

Example 8. If $y = A \cdot \mathrm{sn}(\mu\xi, m)$ is a solution to

$$y'^2 = \mu^2 A^2 - \mu^2(1+m^2)y^2 + \frac{\mu^2 m^2}{A^2} y^4, \tag{30}$$

then $z = A \cdot \mathrm{ns}(\mu\xi, m)$ is a solution to

$$z'^2 = \mu^2 m^2 A^2 - \mu^2(1+m^2)z^2 + \frac{\mu^2}{A^2} z^4. \tag{31}$$

Analogous examples of this type can be obtained with $A \cdot \mathrm{cn}(\mu\xi, m)$ and $A \cdot \mathrm{dn}(\mu\xi, m)$.

Property 3. For equation

$$y'^2 = A^2 - (1+m^2)y^2 + \frac{m^2}{A^2} y^4, \tag{32}$$

besides the solution $y = A \cdot \mathrm{sn}(\xi, m)$, there is another sister solution $y^* = \frac{A}{m}\mathrm{ns}(\xi, m)$. This can be easily verified just by substitution $y^* = \frac{A}{m}\mathrm{ns}(\xi, m)$ into (32).

According to Example 7, $z = A \cdot \mathrm{ns}(\xi, m)$ satisfies

$$z'^2 = m^2 A^2 - (1+m^2)z^2 + \frac{1}{A^2} z^4 \tag{33}$$

and $y^* = \frac{1}{m} z$, thus

$$y^{*\prime 2} = \frac{1}{m^2} z'^2 = A^2 - (1+m^2)y^{*2} + \frac{m^2}{A^2} y^{*4}. \tag{34}$$

That is to say, the equation for y^* is completely identical to that for y.

Corollary: For the equation

$$y'^2 = \mu^2 A^2 - \mu^2(1+m^2)y^2 + \frac{\mu^2 m^2}{A^2} y^4, \tag{35}$$

besides a solution $y = A \cdot \mathrm{sn}(\mu\xi, m)$, there is another sister solution $y^* = \frac{A}{m}\mathrm{ns}(\mu\xi, m)$.

Property 4. For the equation

$$y'^2 = A^2(1-m^2) + (2m^2-1)y^2 - \frac{m^2}{A^2} y^4, \tag{36}$$

besides a solution $y = A \cdot \mathrm{cn}(\xi, m)$, there is another sister solution $y^* = i\frac{\sqrt{1-m^2}}{m} A \cdot \mathrm{nc}(\xi, m)$.

Corollary: For the equation

$$y'^2 = \mu^2 A^2(1-m^2) + \mu^2(2m^2-1)y^2 - \frac{\mu^2 m^2}{A^2} y^4, \tag{37}$$

besides the solution $y = A \cdot \mathrm{cn}(\mu\xi, m)$, there is another sister solution $y^* = i\frac{\sqrt{1-m^2}}{m} A \cdot \mathrm{nc}(\mu\xi, m)$.

Property 5. For the equation

$$y'^2 = -A^2(1-m^2) + (2-m^2)y^2 - \frac{1}{A^2} y^4, \tag{38}$$

besides $y = A \cdot \mathrm{dn}(\xi, m)$, there is another sister solution $y^* = A\sqrt{1-m^2}\,\mathrm{nd}(\xi, m)$.

Corollary: For the equation

$$y'^2 = -\mu^2 A^2(1-m^2) + \mu^2(2-m^2)y^2 - \frac{\mu^2}{A^2} y^4, \tag{39}$$

besides $y = A \cdot \mathrm{dn}(\mu\xi, m)$, there is another sister solution $y^* = A\sqrt{1-m^2}\,\mathrm{nd}(\mu\xi, m)$.

All these statements can be easily checked by substituting the solutions into the respective equations.

4. EODE of the Second Kind

Considering $a_0 = a_4 = 0$ and $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ in (1), then we have

$$y'^2 = a_1 y + a_2 y^2 + a_3 y^3, \tag{40}$$

which may be called EODE of the second kind.

Property 6. If y is a solution of (40), then $z = \frac{1}{y}$ satisfies

$$z'^2 = a_3 z + a_2 z^2 + a_1 z^3. \tag{41}$$

It is obvious, that (41) takes the same form as (40) with a_1 exchanged by a_3. Just as we did in the previous section, this property can be demonstrated by some examples.

Example 9. Equation

$$y'^2 = 4y - 4(1+m^2)y^2 + 4m^2 y^3 \tag{42}$$

has a solution $y = \mathrm{sn}^2(\xi, m)$, and $z = \frac{1}{y}$ satisfies

$$z'^2 = 4m^2 z - 4(1+m^2)z^2 + 4z^3, \tag{43}$$

its solution is $z = \mathrm{ns}^2(\xi, m)$.

Similar to Example 2 and Example 3, analogous examples of this type can be obtained with $\mathrm{cn}^2(\xi, m)$ and $\mathrm{dn}^2(\xi, m)$.

Property 7. If Ay_1, with A a nonzero constant, is a solution of (40), then $z = \frac{A}{y_1}$ satisfies

$$z'^2 = a_3A^2z + a_2z^2 + \frac{a_1}{A^2}z^3. \tag{44}$$

Also this property can be illustrated by examples.

Example 10. It can be proved that $y = A \cdot \mathrm{sn}^2(\xi, m)$ is a solution to

$$y'^2 = 4Ay - 4(1+m^2)y^2 + \frac{4m^2}{A}y^3, \tag{45}$$

by substituting it into (45). Then based on Property 7, we have that $z = A \cdot \mathrm{ns}^2(\xi, m)$ is also a solution to the equation

$$z'^2 = 4m^2Az - 4(1+m^2)z^2 + \frac{4}{A}z^3. \tag{46}$$

Analogous results can be found for $A \cdot \mathrm{cn}^2(\xi, m)$ or $A \cdot \mathrm{dn}^2(\xi, m)$.

Property 8. For the equation

$$y'^2 = 4Ay - 4(1+m^2)y^2 + \frac{4m^2}{A}y^3, \tag{47}$$

besides a solution $y = A \cdot \mathrm{sn}^2(\xi, m)$, there is another sister solution $y^* = \frac{A}{m^2}\mathrm{ns}^2(\xi, m)$.

Corollary: For the equation

$$y'^2 = 4\mu^2Ay - 4\mu^2(1+m^2)y^2 + \frac{4\mu^2m^2}{A}y^3, \tag{48}$$

besides a solution $y = A \cdot \mathrm{sn}^2(\mu\xi, m)$, there is another sister solution $y^* = \frac{A}{m^2}\mathrm{ns}^2(\mu\xi, m)$.

Similar to Property 4, Property 5 and their corresponding corollaries, analogous results can de derived for $A \cdot \mathrm{cn}^2(\mu\xi, m)$, or $A \cdot \mathrm{dn}^2(\mu\xi, m)$.

5. EODE of the Third Kind

Considering $a_4 = 0$ and $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ in (1), then we have

$$y'^2 = a_0 + a_1y + a_2y^2 + a_3y^3, \tag{49}$$

which may be called EODE of the third kind.

Applying the Jacobi elliptic function expansion method [1, 2], it can be easily proved that (49) admits a solution

$$y = A_0 + A_2 \cdot \mathrm{sn}^2(k\xi, m), \tag{50}$$

where A_0, A_2 and k are constants to be shown from boundary conditions.

Property 9. For (49), besides a solution (50), there exists another sister solution

$$y = A_0 + \frac{A_2}{m^2}\mathrm{ns}^2(k\xi, m). \tag{51}$$

Actually, if we suppose that $y_1 = A_0 + v$ and $y_2 = A_0 + w$, $v = A_2 \cdot \mathrm{sn}^2(k\xi, m)$ and $w = A_2 \cdot \mathrm{ns}^2(k\xi, m)/m^2$, then from Example 10, one has

$$v'^2 = 4k^2v - 4k^2(1+m^2)v^2 + \frac{4k^2m^2}{A_2}v^3 \tag{52}$$

and

$$w'^2 = 4k^2w - 4k^2(1+m^2)w^2 + \frac{4k^2m^2}{A_2}w^3. \tag{53}$$

That is to say, v and w satisfy the same equation.

6. EODE of the Fourth Kind

Considering $a_2 = 0$ and $a_0 \neq 0$, $a_1 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$ in (1), then we have

$$y'^2 = a_0 + a_1y + a_3y^3 + a_4y^4, \tag{54}$$

which may be called EODE of the fourth kind.

Similar to (49), the solutions of (54) are

$$y = A_0 + A_1 \cdot \mathrm{sn}(k\xi, m), \tag{55}$$

or

$$y = B_0 + B_1 \cdot \mathrm{cn}(k\xi, m), \tag{56}$$

or

$$y = C_0 + C_1 \cdot \mathrm{dn}(k\xi, m), \tag{57}$$

where A_0, A_1, B_0, B_1, C_0 and C_1 are constants to be determined.

Property 10. For (54), besides the solutions (55), (56), or (57), there exist other solutions

$$y = A_0 + \frac{A_1}{m}\mathrm{ns}(k\xi, m), \tag{58}$$

or

$$y = B_0 + i\frac{\sqrt{1-m^2}}{m}B_1 \cdot \mathrm{nc}(k\xi, m), \tag{59}$$

or

$$y = C_0 + \sqrt{1-m^2}C_1 \cdot \mathrm{nd}(k\xi, m). \quad (60)$$

7. Conclusion and Discussion

In this paper, some properties of the elliptic ordinary differential equation, which can be used to find travelling wave solutions of nonlinear evolution equations, are given. Actually, from these properties, we can obtain more nontrivial information. There are some kinds of symmetries, such as mirror-symmetry in the graphical presentation of solutions given in Example 3, rotational symmetry found in solutions in Example 5. Of course, more solutions may not take these types of symmetries. Just like solutions in Example 1, one is bounded, the other is unbounded (see Fig. 1). Similar behaviors can be found in Example 2, Example 8, Example 9 and Example 10. However, these solutions may take different shapes. There are also cases with two unbounded solutions, such as the solutions given in Example 4 and Example 6. All these results be helpful in studying solutions to nonlinear wave equations.

Acknowledgement

We would like to thank the referees for their careful reading of the manuscript and constructive comments. Many thanks are also due to support from National Natural Science Foundation of China (No. 40305006).

[1] Z. T. Fu, S. K. Liu, S. D. Liu, and Q. Zhao, Phys. Lett. A **290**, 72 (2001).
[2] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, Phys. Lett. A **289**, 69 (2001).
[3] A. V. Porubov, Phys. Lett. A **221**, 391 (1996).
[4] A. V. Porubov and M. G. Velarde, J. Math. Phys. **40**, 884 (1999).
[5] A. V. Porubov and D. F. Parker, Wave Motion **29**, 97 (1999).
[6] Z. T. Fu, S. D. Liu, and S. K. Liu, Commun. Theor. Phys. **39**, 531 (2003).
[7] Z. T. Fu, S. K. Liu, and S. D. Liu, Commun. Theor. Phys. **40**, 285 (2003).
[8] Z. T. Fu, Z. Chen, S. K. Liu, and S. D. Liu, Commun. Theor. Phys. **41**, 675 (2004).
[9] Z. T. Fu, S. D. Liu, S. K. Liu, and Z. Chen, Chaos, Solitons & Fractals **22**, 335 (2004).
[10] S. K. Liu and S. D. Liu, Nonlinear Equations in Physics, Peking University Press, Beijing 2000.
[11] S. K. Liu, Z. T. Fu, S. D. Liu, and Z. G. Wang. Z. Naturforsch. **59a**, 623 (2004).
[12] P.F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists, 2nd Ed., Springer-Verlag, Berlin 1971.
[13] V. Prasolov and Y. Solovyev, Elliptic Functions and Elliptic Integrals, American Mathematical Society, Providence, RI 1997.
[14] Z. X. Wang and D. R. Guo, Special Functions, World Scietific Publishing, Singapore 1989.